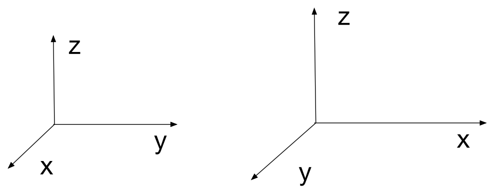
3-D Transformations

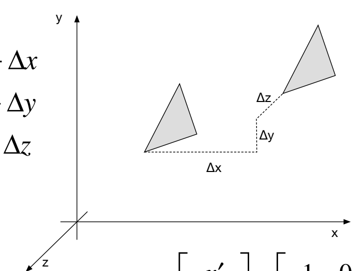


Right-handed Left-handed

4-D homogeneous points

Translation

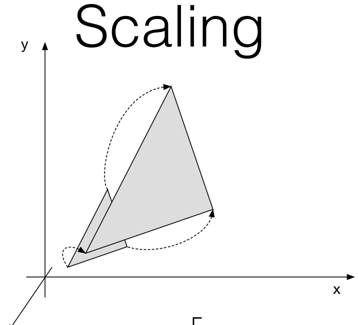
$$x' = x + \Delta x$$
$$y' = y + \Delta y$$
$$z' = z + \Delta z$$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \Delta x \\ 0 & 1 & 0 & \Delta y \\ 0 & 0 & 1 & \Delta z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Scaling

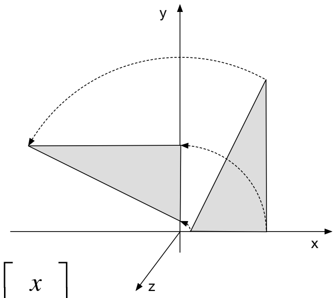
$$x' = s_x \cdot x$$
$$y' = s_y \cdot y$$
$$z' = s_z \cdot z$$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotation (about z-axis)

$$x' = x \cdot \cos\theta - y \cdot \sin\theta$$
$$y' = x \cdot \sin\theta + y \cdot \cos\theta$$
$$z' = z$$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotation (about x-axis)

$$x' = x$$
$$y' = y \cdot \cos\theta - z \cdot \sin\theta$$
$$z' = y \cdot \sin\theta + z \cdot \cos\theta$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotation (about y-axis)

$$x' = x \cdot \cos\theta + z \cdot \sin\theta$$
$$y' = y$$
$$z' = -x \cdot \sin\theta + z \cdot \cos\theta$$

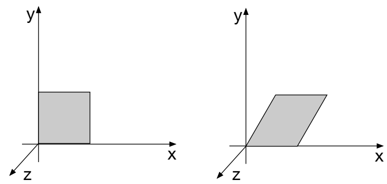
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Shear (along x as a function of y)

Along x: $x' = x + \alpha \cdot y$

$y' = y$

$z' = z$



$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Six Shears

Along x: $x' = x + \alpha \cdot y$, $y' = y$, $z' = z$ $x' = x + \beta \cdot z$, $y' = y$, $z' = z$

Along y: $x' = x$, $y' = \gamma \cdot x + y$, $z' = z$ $x' = x$, $y' = y + \delta \cdot z$, $z' = z$

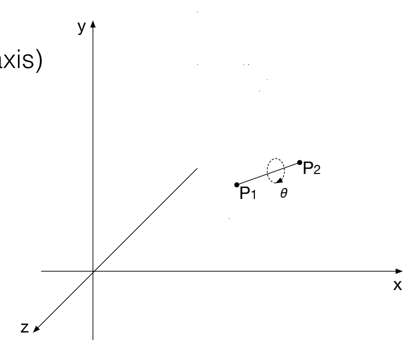
Along z: $x' = x$, $y' = y$, $z' = \varepsilon \cdot x + z$ $x' = x$, $y' = y$, $z' = \varphi \cdot y + z$

$$\begin{bmatrix} 1 & \alpha & \beta & 0 \\ \gamma & 1 & \delta & 0 \\ \varepsilon & \varphi & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotate about arbitrary axis

Given: θ, P_1-P_2 (define axis)

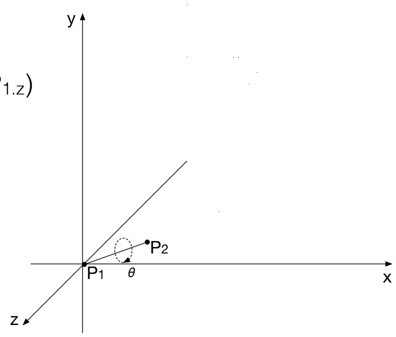
(recall 2D: $T^{-1}RT$)



Rotate about arbitrary axis

1. translate P_1 to origin

Translate($-P_{1.x}$, $-P_{1.y}$, $-P_{1.z}$)

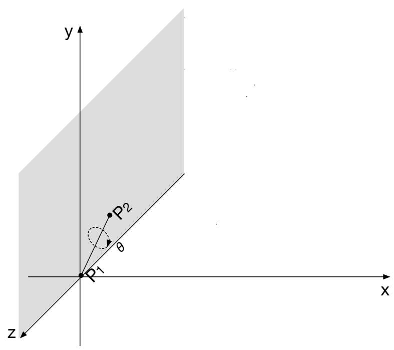


Rotate about arbitrary axis

2. rotate about y-axis so that P_1-P_2 lies in the yz plane

Let $dx = P_{2.x} - P_{1.x}$
$dy = P_{2.y} - P_{1.y}$
$dz = P_{2.z} - P_{1.z}$

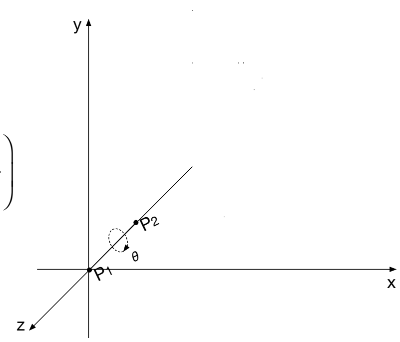
$$\theta_y = \arccos\left(\frac{-dz}{\sqrt{dx^2 + dz^2}}\right)$$



Rotate about arbitrary axis

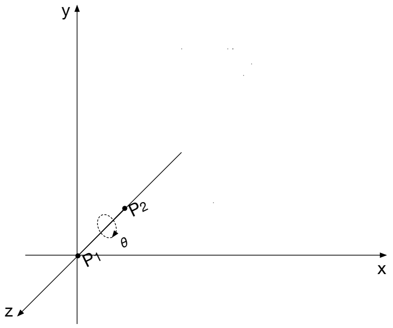
3. rotate about the x-axis so that P_1-P_2 lies on the x-axis

$$\theta_x = \arcsin\left(\frac{dy}{\sqrt{dx^2 + dy^2 + dz^2}}\right)$$



Rotate about arbitrary axis

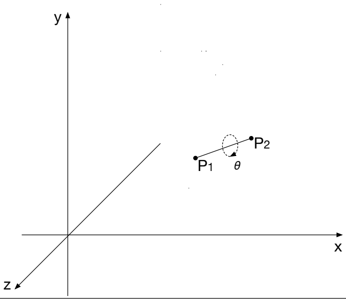
4. rotate about the z-axis using θ



Rotate about arbitrary axis

1. translate P_1 to origin
2. rotate about y-axis so that P_1-P_2 lies in the yz plane
3. rotate about the x-axis so that P_1-P_2 lies on the x-axis
4. rotate about z axis using θ
5. $3)^{-1}$
6. $2)^{-1}$
7. $1)^{-1}$

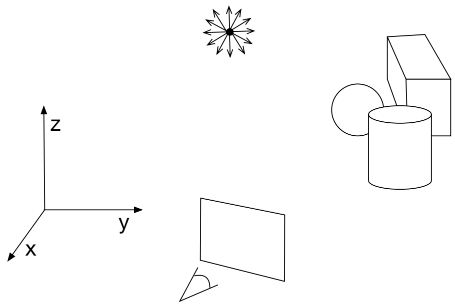
$T_7 \cdot R_6 \cdot R_5 \cdot R_4 \cdot R_3 \cdot R_2 \cdot T_1$



Coordinate Systems

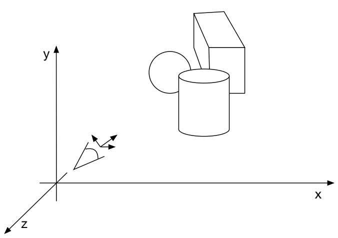
- Model/Object Coordinate System
  - where you define object
- World Coordinate System
  - where objects are placed relative to each other
- Eye Coordinate System
  - Eye at origin, looking along z-axis
- Screen Coordinate System
  - device specific coordinates

Coordinate Systems



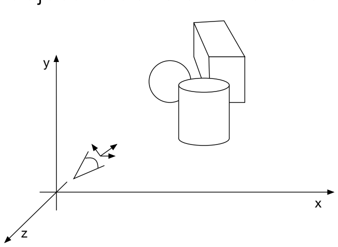
Eye Coordinate System

- eye is at origin
- eye is looking along z axis (l.h.s.?)
- x-axis is horizontal
- y-axis is vertical



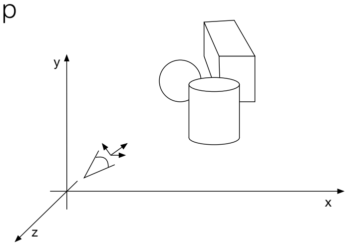
Changing Coord Sys

- every coord system has a frame of reference (origin, unit basis vectors)
- Going from one coord sys to another can be done with a transformation matrix
- transform object vs transform frame of reference

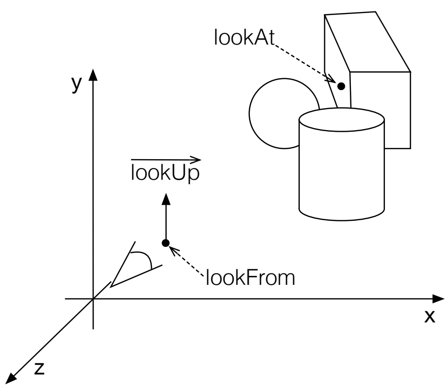


Viewing Transformation

- world coordinate system -> eye coordinate system
- eye at origin, looking along z-axis (l.h.s.)
- how to specify?
  - lookFrom, lookAt, $\overrightarrow{\text{lookUp}}$

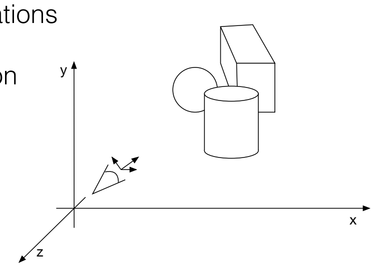


Viewing Transformation



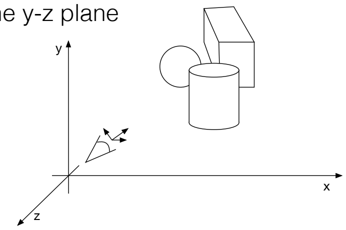
Computing Viewing Transformation

- 2 ways:
  - Series of transformations
  - Direct matrix creation



Series of Transformations

- given: lookFrom, lookAt, $\overrightarrow{\text{lookUp}}$
- translate lookFrom to origin
- rotate such that lookAt is on the negative z-axis (2 rotations)
- rotate so that $\overrightarrow{\text{lookUp}}$ is on the y-z plane
- RHS->LHS



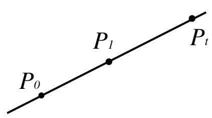
Aside: Vectors

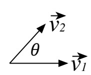
$$P_t = P_0 + t \cdot (P_1 - P_0)$$

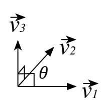
$$P_t = P_0 + t \cdot \vec{v}$$

$$\vec{v}_1 \bullet \vec{v}_2 = \|\vec{v}_1\| \cdot \|\vec{v}_2\| \cdot \cos(\theta)$$

$$\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$$
$$\rightarrow \|\vec{v}_3\| = \|\vec{v}_1\| \cdot \|\vec{v}_2\| \cdot \sin(\theta)$$
$$\rightarrow \vec{v}_1 \bullet \vec{v}_3 = 0$$
$$\rightarrow \vec{v}_2 \bullet \vec{v}_3 = 0$$





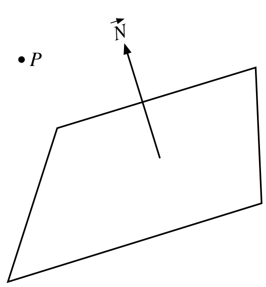


Aside: Planes

$$a \cdot x + b \cdot y + c \cdot z + d = 0$$

$$\vec{N} = (a,b,c)$$
$$\vec{N} \cdot (x,y,z) + d = 0$$

$$\vec{N} \cdot P + d = ?$$



If the $\vec{N}$ is of unit length, result is distance from plane

Defining Planes

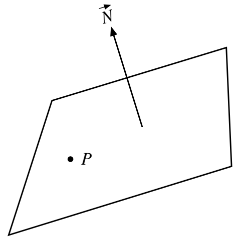
How?

- 3 points or
- 1 normal and 1 point somewhere on the plane

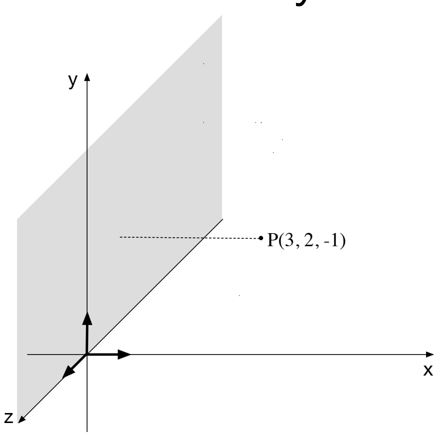
$$a \cdot x + b \cdot y + c \cdot z + d = 0$$

$$\vec{N} = (a, b, c)$$
$$\vec{N} \cdot (x, y, z) + d = 0$$

$$d = -\vec{N} \cdot P$$



Planes & Coord Systems



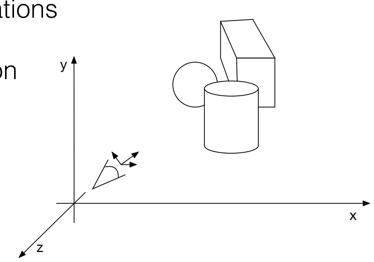
Planes and Matrices

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x & d_x \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$x' = a_x \cdot x + b_x \cdot y + c_x \cdot z + d_x$$

Computing Viewing Transformation

- 2 ways:
  - Series of transformations
  - Direct matrix creation



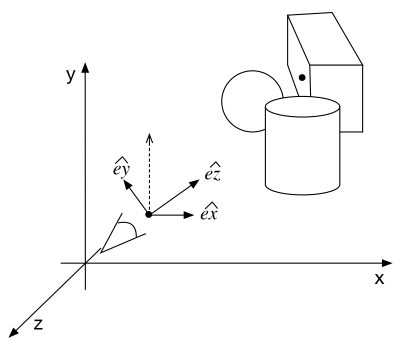
Viewing Transformation

given: $lookFrom, lookAt, \overrightarrow{lookUp}$

$$\widehat{ez} = \frac{lookAt - lookFrom}{\|lookAt - lookFrom\|}$$

$$\widehat{ex} = \frac{\widehat{ez} \times \overrightarrow{lookUp}}{\|\widehat{ez} \times \overrightarrow{lookUp}\|}$$

$$\widehat{ey} = \widehat{ex} \times \widehat{ez}$$



Direct Matrix Creation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} ex_x & ex_y & ex_z & d_x \\ ey_x & ey_y & ey_z & d_y \\ ez_x & ez_y & ez_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$d_x = -\widehat{ex} \bullet lookFrom$$

$$d_y = -\widehat{ey} \bullet lookFrom$$

$$d_z = -\widehat{ez} \bullet lookFrom$$